Design of Impedance Transformers by the Method of Least Squares

Homayoon Oraizi, *Member, IEEE*

Abstract—**The method of least squares is applied to the theory of small reflections of transmission lines to develop numerical algorithms for the design of stepline and tapered line impedance transformers to match two impedances over a frequency band. The transformer characteristic impedance function is expanded by polynomials, pulse functions, approximate operators, and piecewise linear functions to construct an error function for the input reflection coefficient which, after minimization, gives the line impedance and length. The computer programs could be used to design a transformer under the specified conditions and then to optimize the design under the constraints of a problem.**

I. Introduction

MATCHING networks are frequently required at the load end of a transmission line and at its generator side. Impedance transformers are also required at various points in a microwave network. Impedance matching techniques are broadly divided into two types [1]:

1) maximum power transfer or conjugate matching whereby a load impedance (or a transmission line input impedance) is set equal to the complex conjugate of the generator impedance;
2) reflectionless or Zo matching whereby a load impedance is set equal to the characteristic impedance of a transmission line. This results in a reflection coefficient of zero at the load and an SWR on the line equal to unity.

Transformers are, in general, required to match two general complex impedances (not necessarily constant) over a frequency band. Several procedures are available for the design of tapers and steplines (corresponding to high pass and band pass filters, respectively) to achieve reflectionless transitions between two different line impedances over a frequency band. Excellent discussions of impedance transformation and matching are available in several books [1]–[6] that refer to the relevant literature [7].

Stepline transformers are made of a cascade of uniform sections of transmission lines such as quarterwave, quarterwave multisection, binomial, and Chebyshev transformers [2], [8]–[11]. Step discontinuities, however, introduce junction impedances (such as change in width of microstrips and striplines), which should be accounted for in the design of stepline transformers. Tapered line transformers are made of a nonuniform section of a transmission line. The taper is usually named after the function of distance along the line which describes the characteristic impedance or reflection coefficient such as exponential, triangular, and Chebyshev tapers [2], [13]–[15].

As is seen, the theory of stepline and tapered line transformers is quite involved. We recall that the input reflection coefficient of a taper is governed by a Riccati differential equation, however, little effort to date has been spent for the development of numerical methods for the synthesis of nonuniform transformers.

In this paper, a numerical procedure is developed to synthesize stepline and tapered line matching sections by the method of least squares (MLS). The theory of small reflections is assumed, and the input reflection coefficient of the matching section is minimized to determine its characteristic impedance variation. The least squares numerical procedure is first developed. The computer implementation is then presented. The results of the transformer designs agree well with the published data obtained by other design methods. Several interesting behaviors of the stepline and tapered line matching sections are revealed by the computer programs run for various examples of impedance matching.

Besides developing an easily implemented numerical procedure for the design of stepline and tapered line transformers of any length over a frequency band, the solution of this problem by the method of least squares illustrates the power and applicability of this method for the solution of electromagnetic problems.

II. Numerical Procedure

The theory of small reflections of the nonuniform lines is applied for the development of a numerical algorithm for the synthesis of a stepline or a taper to match two impedances over a frequency band. Consider a tapered line with characteristic impedance $Z(z)$ as a function of distance connecting a voltage source with internal impedance Z_g to a load impedance Z_L as in Fig. 1. Assume that the characteristic impedance of the input line is equal to the internal impedance of the source ($Z_g = Z_c$), which is resistive and constant in the desired frequency band. All impedances are normalized with respect to the input line characteristic impedance as

$$\overline{Z}_c = 1, \overline{Z}(z) = Z(z)/Z_c, \quad \text{and} \quad \overline{Z}_L = \overline{Z}_L/Z_c.$$

Now the reflection coefficient at the tapered line input is [2]

$$\Gamma_i = \frac{1}{2}\int_0^L e^{-j2\beta z}\frac{d}{dz}\ln\overline{Z}(z)\,dz \tag{1}$$

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The author is with the Faculty of the Department of Electrical Engineering, Iran University of Science and Technology, Narmak, Tehran, 16844 Iran.
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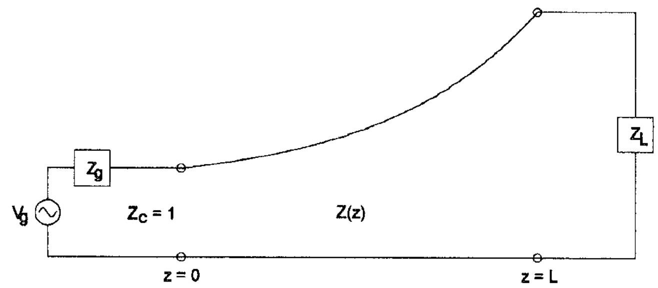

Fig 1 A tapered line connecting a voltage source to a load.

$$= \frac{1}{2}e^{-j2\beta L}\ln \overline{Z}(L) + j\beta\int_0^L e^{-j2\beta z}\ln \overline{Z}(z)\,dz. \quad (2)$$

The latter expression is obtained through integration by parts. The boundary conditions at the two ends of the line are

$$\overline{Z}(0) = \overline{Z}_c = 1,\ z = 0$$
$$\overline{Z}(L) = \overline{Z}_L,\ z = L \quad (3)$$

where L is the length of the nonuniform line.

To develop an algorithm for the determination of the stepline and tapered line shape, the functions $\overline{Z}(z)$, ln $\overline{Z}(z)$ or d/dz ln $\overline{Z}(z)$ are represented by a polynomial, pulse function or step approximation, triangle function or piecewise linear approximation, and an approximate operator [16]. An error function is then constructed for Γ_i and minimized to obtain the coefficients of the polynomial or the amplitudes of the expansion functions.

A. Polynomial Expansion of $\frac{d}{dz}\ln\overline{Z}$

Assume that the function $d/dz \ln \overline{Z}$ can be approximated by a polynomial of degree N, from which ln $\overline{Z}$ and $\overline{Z}$ may be obtained

$$\frac{d}{dz}\ln\overline{Z}(z) = \sum_{n=0}^{N} a_n z^n \quad (4)$$

$$\overline{Z}(z) = \exp\left[\sum_{n=0}^{N}\frac{1}{n+1}a_n z^{n+1} + c\right] \quad (5)$$

where c is a constant. We invoke the boundary conditions (3) to determine $c = 0$ and a_o in terms of the other coefficients of the polynomial

$$a_o = \frac{1}{L}\ln\overline{Z}_L - \sum_{n=1}^{N}\frac{a_n}{n+1}L^n. \quad (6)$$

Now, a_o from (6) is replaced in (4) and integrated or a_o is substituted in (5) to obtain an expression for ln $\overline{Z}$

$$\ln\overline{Z} = \left(\frac{1}{L}\ln\overline{Z}_L - \sum_{n=1}^{N}\frac{a_n}{n+1}L^n\right)z + \sum_{n=1}^{N}\frac{a_n}{n+1}z^{n+1}. \quad (7)$$

Equation (7) is substituted in (1)

$$\Gamma_i = \frac{1}{2L}\ln\overline{Z}_L\int_0^L e^{-j2\beta z}\,dz + \frac{1}{2}\sum_{n=1}^{N}a_n \cdot\left[\int_0^L z^n e^{-j2\beta z}\,dz - \frac{L^n}{n+1}\int_0^L e^{-j2\beta z}\,dz\right]. \quad (8)$$

The second integral in (8) is integrated by parts successively or obtained from tables of integrals [17]. Then

$$\Gamma_i = \frac{1}{2}\ln(\overline{Z}_L)\frac{\sin\beta L}{\beta L}e^{-j\beta L} - \frac{1}{2}\sum_{n=1}^{N}a_n L^{n+1} \cdot\left[\frac{n!e^{-j2\beta L}}{j2\beta L}\sum_{i=0}^{n}\frac{1}{(n-i)!(j2\beta L)^i} - \frac{n!}{(j2\beta L)^{n+1}} + \frac{1}{n+1}\frac{\sin\beta L}{\beta L}e^{-j\beta L}\right]. \quad (9)$$

Note that Γ_i is a linear function of a_n, which should be determined.

Usually, an impedance matching is required over a frequency band, which is divided into K discrete frequencies. Now, the expression for Γ_i in (9) can be written in a concise form as [16]

$$\Gamma_{ik} = t_k + \sum_{n=1}^{N} l_{nk}a_n \quad (10)$$

where t_k and l_k are defined as in (9). The subscript k indicates the kth frequency and the quantities β and $\overline{Z}_L$ should be substituted by β_k and $\overline{Z}_{Lk}$, respectively. Then, an error function is constructed as

$$\epsilon = \sum_{k=1}^{K}\Gamma_{ik}\Gamma_{ik}^* = \sum_{k=1}^{K}\left(t_k + \sum_{n=1}^{N}l_{nk}a_n\right)\left(t_k^* + \sum_{n=1}^{N}l_{nk}^*a_n\right). \quad (11)$$

The error function is constructed in such a way that its minimum point gives the minimization of $|\Gamma_i|^2$ and SWR [2, p. 299]. To minimize the error function, we take the derivatives of ϵ with respect to a_n and equate to zero to obtain an expression for a_n

$$[a_n] = \left[\mathrm{Re}\sum_{k=1}^{K}l_{nk}l_{mk}^*\right]^{-1}\left[-\mathrm{Re}\sum_{k=1}^{K}l_{mk}^*t_k\right]. \quad (12)$$

Therefore, determination of the polynomial coefficients leads to a mere matrix inversion.

The error function is a nonquadratic function of the taper length (L). Therefore, its minimization with respect to L is performed separately.

The derivative of the error function with respect to L is

$$\frac{\partial\epsilon}{\partial L} = 2\,\mathrm{Re}\sum_{k=1}^{K}\left(\frac{\partial t_k}{\partial L} + \sum_{n=1}^{N}a_n\frac{\partial l_{nk}}{\partial L}\right)\left(t_k^* + \sum_{n=1}^{N}l_{nk}^*a_n\right) \quad (13)$$

$$\frac{\partial t_k}{\partial L} = \frac{1}{2L}\ln\overline{Z}_{Lk}\left[\cos\beta_k L - (1 + j\beta_k L)\frac{\sin\beta_k L}{\beta_k L}\right]e^{-j\beta_k L} \quad (14)$$

$$\frac{\partial l_{nk}}{\partial L} = \frac{nL^n}{n+1}\left[\cos\beta_k L - (1 + j\beta_k L)\frac{\sin\beta_k L}{\beta_k L}\right]e^{-j\beta_k L}. \quad (15)$$

Minimization of ϵ for L is performed by any one of the usual routines of linear search such as the steepest descent

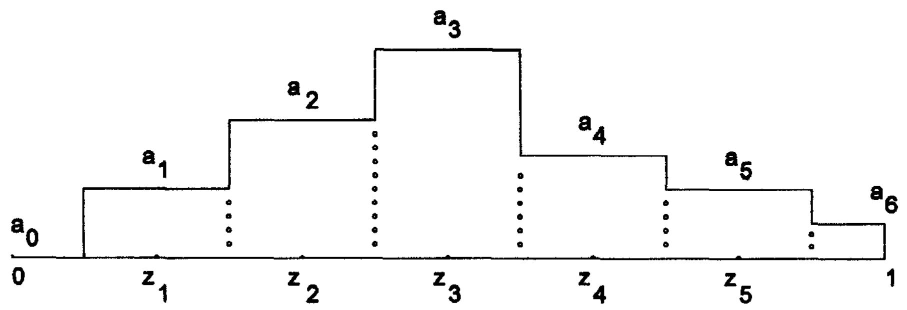

Fig. 2. Approximation of $\ln \overline{Z}(z)$ by PE.

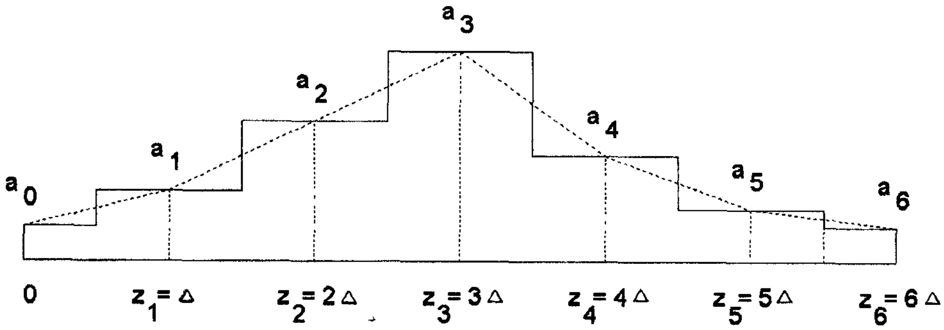

Fig. 3. Representation of $\overline{Z}(z)$ by impulse functions employing AO for $d/dz \ln \overline{Z}$.

and interval halving [18]. Once the taper length is obtained, the new values of the polynomial coefficients may be determined by (12). In many cases, however, the length of the taper is constrained. Finally, the characteristic impedance function of the taper is determined by (7) once the coefficients a_n are calculated.

Approximation of $\ln \overline{Z}(z)$ by a polynomial function leads to the same results obtained above and could serve as a check of their accuracy.

For an example, suppose the polynomial in (4) has only one term a_o. Invoking the boundary conditions (3) leads to

$$\overline{Z}(z) = \exp\left(\frac{z}{L} \ln \overline{Z}_L\right).$$

The reflection coefficient at the line input according to (1) is

$$\Gamma_i = \frac{1}{2} \ln \overline{Z}_L \frac{\sin \beta L}{\beta L} e^{-j\beta L}$$

and the error function is $\epsilon = |\Gamma_i|^2$. The derivative of ϵ with respect to L is

$$\frac{\partial \epsilon}{\partial L} = \frac{1}{2L} (\ln \overline{Z}_L)^2 \frac{\sin^2 \beta L}{(\beta L)^2} (\beta L \cot \beta L - 1).$$

The minima of ϵ occur at $\beta L = n\pi$ for $n = 1, 2, 3, \cdots$

$$L = 2n\frac{\lambda}{4}, \; n = 1, 2, 3, \cdots.$$

The maxima of ϵ occur for $\tan \beta L = \beta L$ which may be determined graphically. For large values of βL we have

$$\beta L = (2n+1)\pi/2 \quad \text{and} \quad L = (2n+1)\lambda/4$$
$$\text{for } n = 0, 1, 2, \cdots.$$

The above relations in this example may also be obtained from (7), (9), (11), (13)–(15). Compare this result with [2, p. 372].

B. Approximation of $\ln \overline{Z}(z)$ by Pulse Functions

The multisection line of length L is assumed to have $N+1$ sections as in Fig. 2. The length of each section is $\Delta = L/(N+1)$. The midpoints of the sections are at $z_n = n\Delta$ for $n = 1, 2, \cdots, N$. The pulse in the nth section is defined as

$$P_n(z - z_n) = \begin{cases} 1 & |z - z_n| < \frac{\Delta}{2} \\ 0 & |z - z_n| > \frac{\Delta}{2}. \end{cases} \tag{16}$$

Now the function $\ln \overline{Z}(z)$ in the interval $0 \leq z \leq L$ on the multisection line is expanded as pulse functions

$$\ln \overline{Z}(z) = \sum_{n=0}^{N+1} a_n P_n(z - z_n) \tag{17}$$

where a_n are the pulse amplitudes. The pulses P_o and P_{N+1} are defined in the intervals $0 \leq z < \Delta/2$ and $L - \Delta/2 < z < L$, respectively. All impedances are normalized with respect to the input impedance. Invoking the boundary conditions in (3) leads to $a_o = 0$ and $a_{N+1} = \ln \overline{Z}_L$, which are replaced in (17). To determine $d/dz \ln [\overline{Z}(z)]$, the derivatives of the pulse functions should be taken, which lead to impulse functions at $n\Delta \pm \Delta/2$. Substituting this function into (1) and performing the resulting integrations noting the sampling property of the impulse functions, we finally have

$$\Gamma_i = j \sin(\beta\Delta) \sum_{n=1}^{N} a_n \exp(-j2n\beta\Delta) + \frac{1}{2} \ln \overline{Z}_L \exp[-j(2N+1)\beta\Delta]. \tag{18}$$

Equation (18) can also be derived using (2).

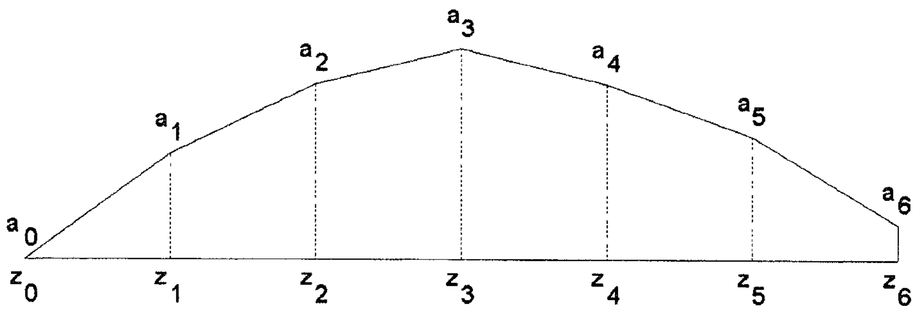

Fig. 4. Piecewise linear approximation (PLA) of ln $\overline{Z}(z)$.

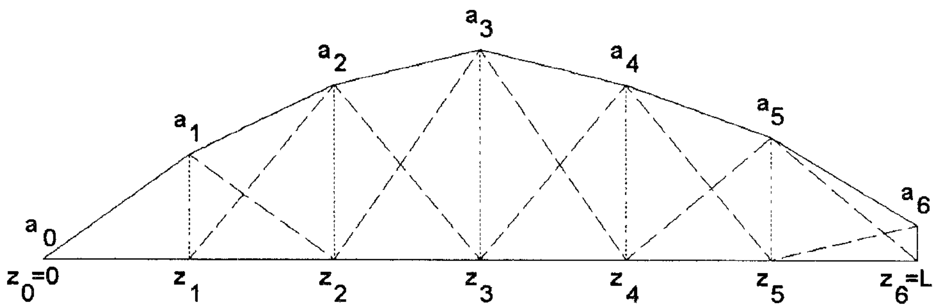

Fig. 5 Approximation of ln $\overline{Z}(z)$ by triangle functions PLA

TABLE I
EXAMPLES OF IMPEDANCE TRANSFORMER DESIGN

Meth.	Fig.	L_{in} (cm)	N	F_l (GHz)	F_u (GHz)	K No.Freq.	Z_{in} (Ω)	Z_L (Ω)	ε_{in} initial error	L_f (cm)	ε_f final error
PA	6	5.0	2	1.99	4.01	20	1	2	0.100534	10.8	0.006717
PE	7	7.5	7	1	5	20	1	5	0.557775	19.7	0.010562
AO	8	7.5	7	1	5	20	1	5	0.526697	20.7	0.004655
PLE	9	7.5	8	1	5	20	1	5	0.526697	20.7	0.004655
PLE	10	10.0	5	1	2	20	1	2	0.023877	24.7	0.000064
PE	11	7.5	4	1	5	20	1	10	1.743317	12.0	0.420956
PLE	12	7.5	4	1	5	20	1	10	1.522610	33.0	0.043228

Impedance transformers are usually designed over a frequency band. Therefore, to use (12) for the determination of the pulse amplitudes, we require the following

$$\mathrm{Re}\left(\sum_{k=1}^{K} l_{mk}^{*} l_{nk}\right) = \sum_{k=1}^{K} \sin^2(\beta_k\Delta) \cdot \cos[2(m-n)\beta_k\Delta] \tag{19}$$

$$\mathrm{Re}\left(\sum_{k=1}^{K} l_{mk}^{*} t_k\right) = \frac{1}{2}\sum_{k=1}^{K} \ln \overline{Z}_L \sin(\beta_k\Delta) \cdot \sin[(2m-2N-1)\beta_k\Delta]. \tag{20}$$

For the minimization of the error function with respect to the line length we use (13) with the following

$$\frac{\partial t_k}{\partial L} = -j\frac{2N+1}{2(N+1)}\beta_k \ln \overline{Z}_L \cdot \exp[-j(2N+1)\beta_k\Delta] \tag{21}$$

$$\frac{\partial l_{nk}}{\partial L} = \frac{\beta_k}{N+1}[2n\sin(\beta_k\Delta) + j\cos(\beta_k\Delta)] \cdot \exp(-j2n\beta_k\Delta). \tag{22}$$

The characteristic impedance of the multisection line in terms of the pulse amplitudes is

$$\overline{Z}(z) = \sum_{n=1}^{N} e^{a_n} P_n(z-n\Delta) + \overline{Z}_L P_{N+1}(z-L). \tag{23}$$

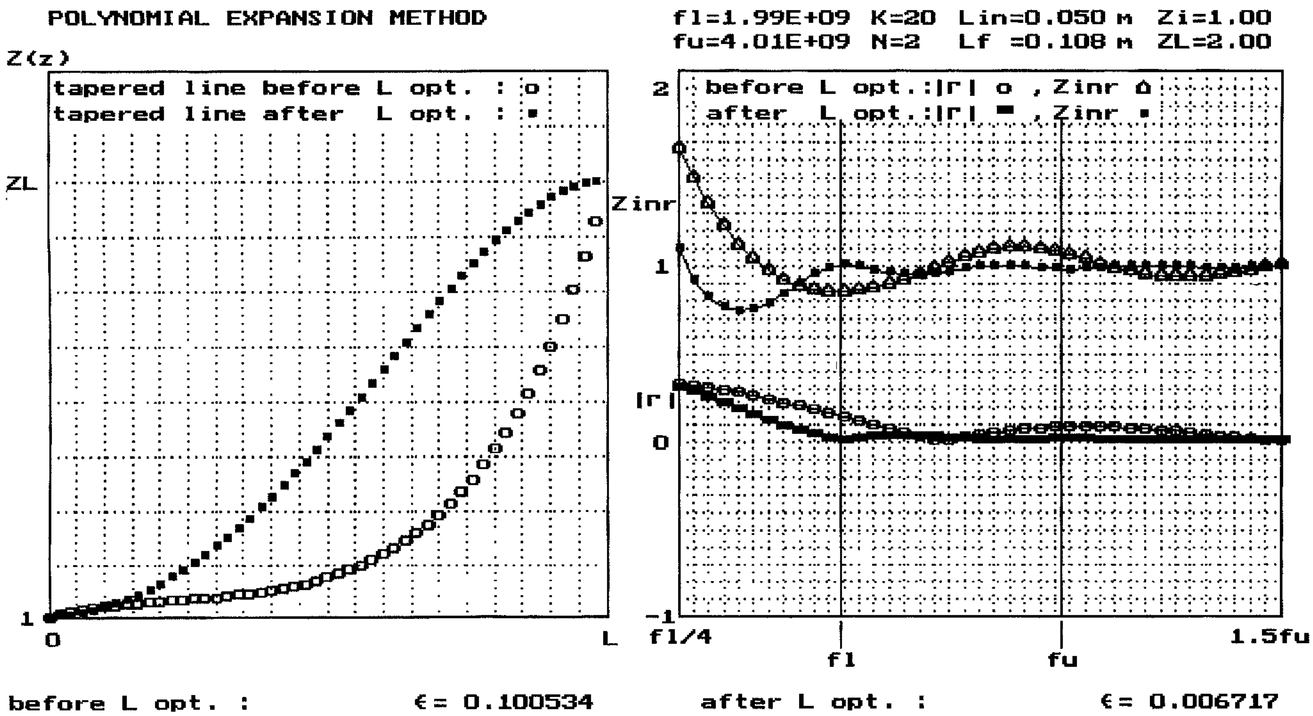

Fig. 6. Tapered line transformer design by polynomial expansion before and after length optimization.

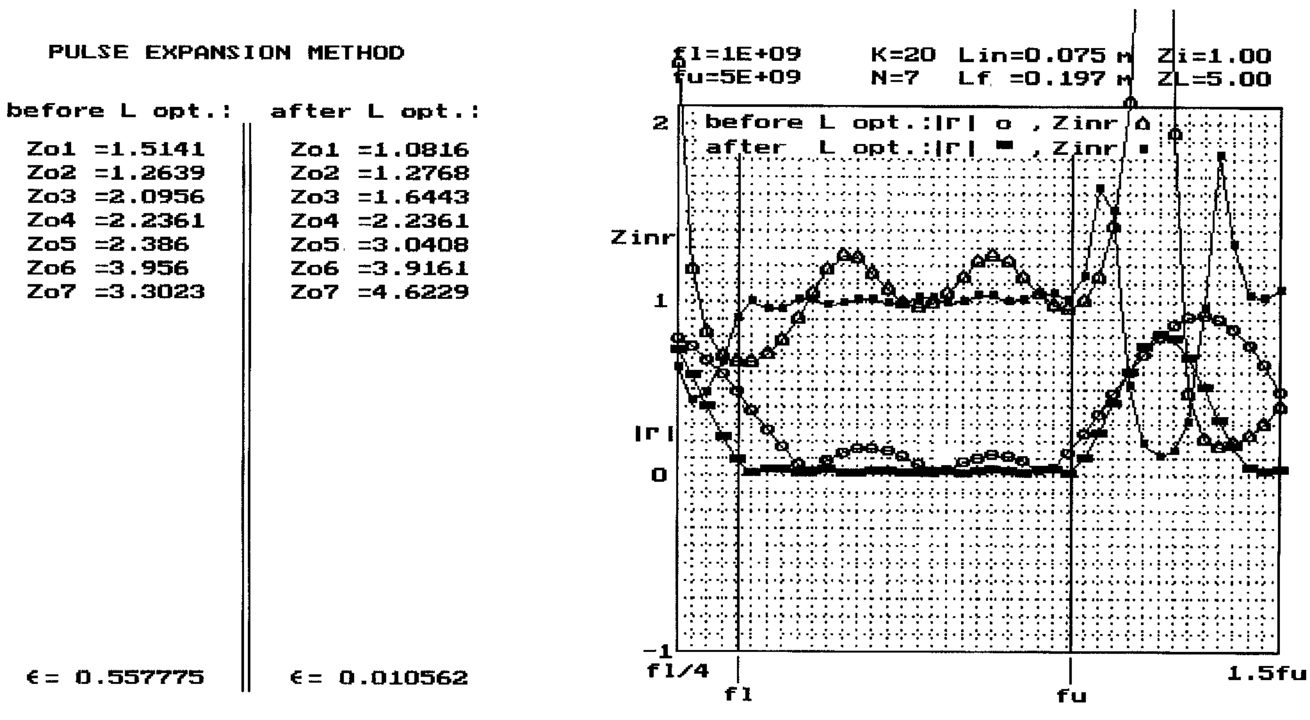

Fig. 7. Stepline design by PE before and after length optimization.

For example, suppose that a line of length L is divided into two sections. Therefore, $N = 1$, $\Delta = L/2$, and $z_1 = L/2$. The pulse functions are $P_o(z) = 1$ for $0 \le z < L/4$, $P_1(z - L/2) = 1$ for $L/4 < z < 3L/4$, and $P_2(z - L) = 1$ for $3L/4 < z \le L$. The line input reflection coefficient from (18) and the corresponding error function from (11) are

$$\Gamma_i = j\sin\left(\frac{\beta L}{2}\right)e^{-j\beta L}a_1 + \frac{1}{2}\ln\overline{Z}_L e^{-j3\beta L/2}$$

$$\epsilon = \sin^2\frac{\beta L}{2}a_1^2 - \ln\overline{Z}_L\sin^2\frac{\beta L}{2}a_1 + \frac{1}{4}\ln^2\overline{Z}_L.$$

Taking the derivative of ϵ with respect to a_1 and equating to zero gives $a_1 = \ln\sqrt{\overline{Z}_L}$. The characteristic impedance of the line by (23), the input reflection coefficient and the error function for this value of a_1 are

$$\overline{Z}(z) = \sqrt{\overline{Z}_L}P_1\left(z - \frac{L}{2}\right) + \overline{Z}_L P_2(z - L)$$

APPROXIMATE OPERATOR METHOD

before L opt.:	after L opt.:
Zo1 =1.5474	Zo1 =1.0828
Zo2 =1.248	Zo2 =1.2782
Zo3 =2.1257	Zo3 =1.6455
Zo4 =2.2361	Zo4 =2.2361
Zo5 =2.3522	Zo5 =3.0387
Zo6 =4.0063	Zo6 =3.9118
Zo7 =3.2311	Zo7 =4.6177
ϵ= 0.526697	ϵ= 0.004655

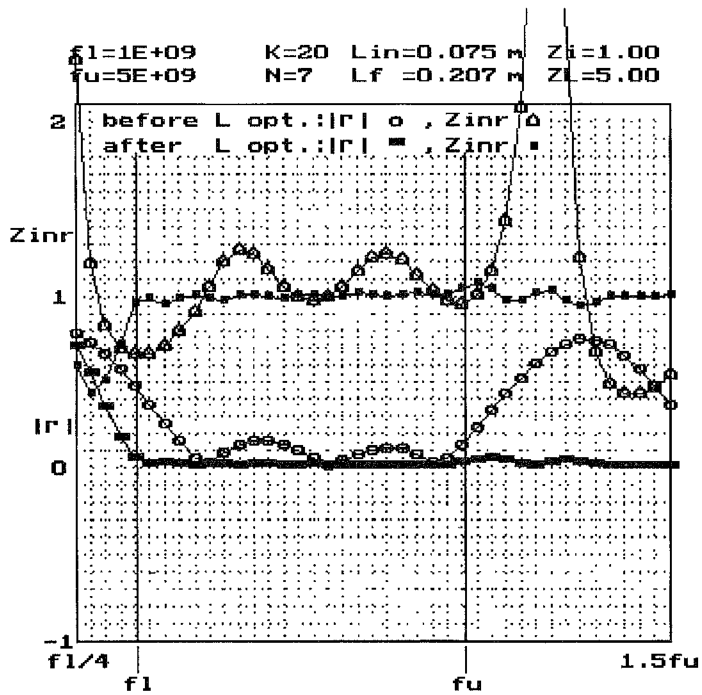

Fig. 8. Stepline design by AO before and after length optimization.

$$\Gamma_i = \frac{1}{2}\ln \overline{Z}_L \cos\left(\frac{\beta L}{2}\right) e^{-j\beta L}$$

$$\epsilon = \frac{1}{4}\ln^2 \overline{Z}_L \cos^2 \frac{\beta L}{2}.$$

The error function becomes zero for odd multiples of a half wavelength $L = (2n+1)L/2$, $n = 0, 1, 2, \cdots$. Note that in this formulation one section of length $\lambda/8$ with impedance $\overline{Z}_c = 1$ is placed at the input end of the line and one $\lambda/8$ line of impedance is placed at the output end. Therefore, a quarterwave line of impedance $\sqrt{\overline{Z}_L}$ matches the two impedances $\overline{Z}_L$ and 1. For the design of an N section quarterwave transformer by this method, the line length should be $L = (N+1)\lambda/4$ or $\Delta = \lambda/4$.

C. Approximate Operator for $\frac{d}{dz}\ln \overline{Z}(z)$

The line of length L is divided into $N+1$ sections. The width of each section is $\Delta = L/(N+1)$. The distance of each division point from the input end is $z_n = n\Delta$ as in Fig. 3. The characteristic impedance $\overline{Z}(z)$ is represented as a series of impulse functions in the interval $0 \le z \le L$

$$\overline{Z}(z) = \sum_{n=0}^{N} a_n \delta(z - n\Delta). \tag{24}$$

The boundary conditions (3) give $a_o = 1$ for $0 \le z < \Delta/2$ and $a_{N+1} = \overline{Z}_L$ for $L - \Delta/2 < z \le L$. We use the approximate operator for the derivative of $\overline{Z}(z)$ at the midpoint of each section

$$\frac{d}{dz}\ln \overline{Z}(z)\,|_{z=(n-1/2)\Delta} \approx \frac{1}{\Delta}[\ln \overline{Z}(n\Delta) - \ln \overline{Z}(n\Delta - \Delta)]$$

$$\approx \frac{1}{\Delta}\ln\left[\frac{\overline{Z}(n\Delta)}{\overline{Z}(n\Delta - \Delta)}\right] = \frac{1}{\Delta}\ln\left(\frac{a_n}{a_{n-1}}\right), \; (n-1)\Delta < z < n\Delta. \tag{25}$$

The approximate expression (25) is inserted in (1) and the resulting simple integration is carried out to obtain the input reflection coefficient

$$\Gamma_i = \frac{\sin\beta\Delta}{\beta\Delta}\left[j\sin\beta\Delta\sum_{n=1}^{N}\ln(a_n)e^{-j2n\beta\Delta} + \frac{1}{2}\ln(\overline{Z}_L)e^{-j(2N+1)\beta\Delta}\right]. \tag{26}$$

We use (12) to determine ln a_n with the following

$$\mathrm{Re}\left[\sum_{k=1}^{K} l_{mk}^* l_{nk}\right] = \sum_{k=1}^{K}\frac{\sin^4(\beta_k\Delta)}{(\beta_k\Delta)^2} \cdot \cos[2(m-n)\beta_k\Delta] \tag{27}$$

$$\mathrm{Re}\left[\sum_{k=1}^{K} l_{mk}^* t_k\right] = \frac{1}{2}\sum_{k=1}^{K}\ln \overline{Z}_L \frac{\sin^3(\beta_k\Delta)}{(\beta_k\Delta)^2} \cdot \sin[(2m-2N-1)\beta_k\Delta]. \tag{28}$$

Minimization of the error function with respect to the line length is carried out by (13) using the following

$$\frac{\partial l_{nk}}{\partial L} = j\frac{1}{L}e^{-j2n\beta_k\Delta} \cdot \left[\sin(2\beta_k\Delta) - (1 + j2n\beta_k\Delta)\frac{\sin^2(\beta_k\Delta)}{\beta_k\Delta}\right] \tag{29}$$

$$\frac{\partial t_k}{\partial L} = \frac{1}{2L}\ln \overline{Z}_L e^{-j(2N+1)\beta_k\Delta} \cdot \left[e^{-j\beta_k\Delta} - (1 + j2N\beta_k\Delta)\frac{\sin(\beta_k\Delta)}{\beta_k\Delta}\Big)\right]. \tag{30}$$

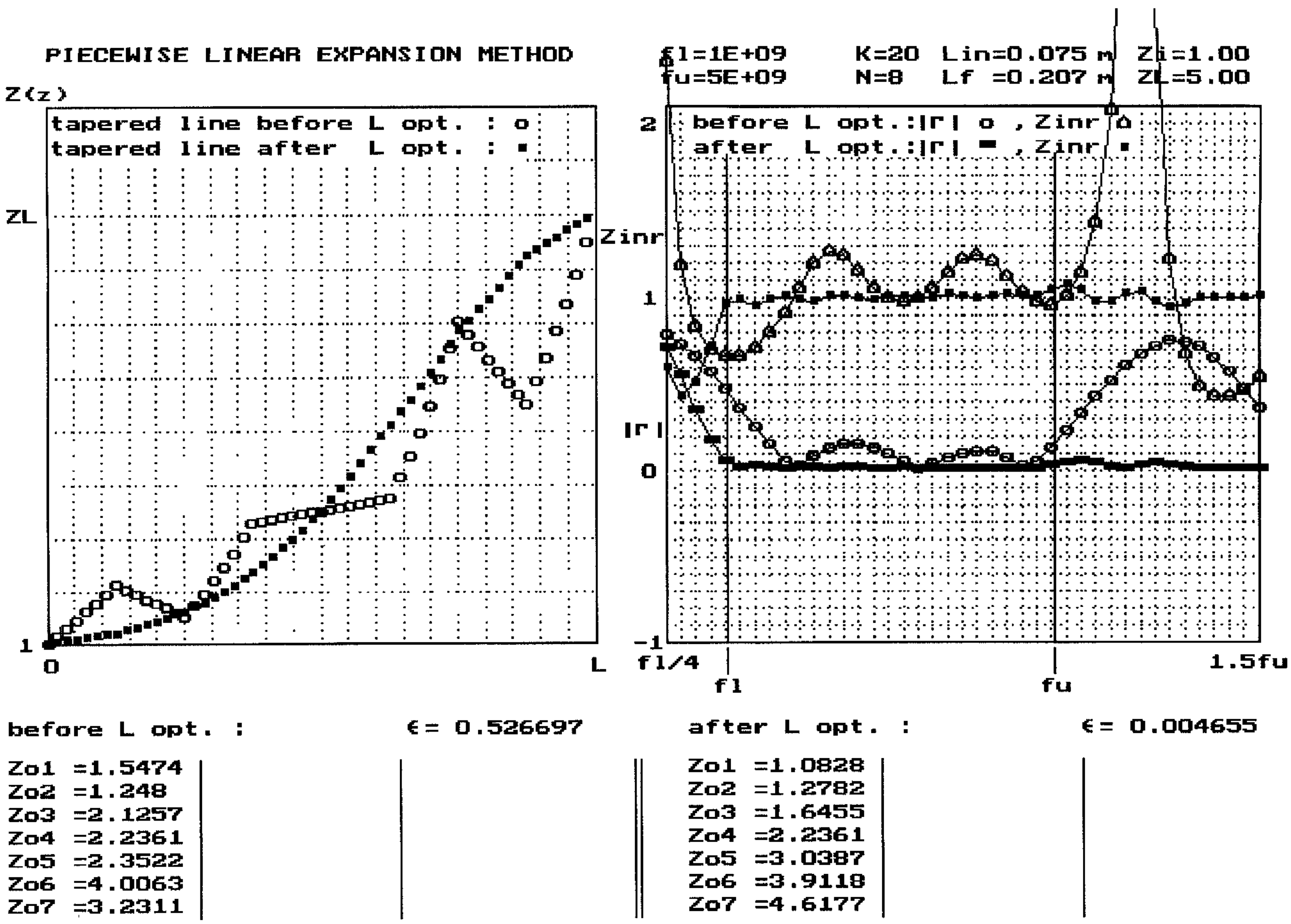

Fig 9. Tapered line design by PLA before and after length optimization.

For example, suppose that the line length L is divided into two sections of width $\Delta = L/2$ and $N = 1$. Therefore

$$\Gamma_i = \frac{\sin\beta\Delta}{\beta\Delta} e^{-j2\beta\Delta}\left[j\sin\beta\Delta \ln a_1 + \frac{1}{2}\ln \overline{Z}_L e^{-j\beta\Delta}\right]$$

$$\epsilon = \frac{\sin^4\beta\Delta}{(\beta\Delta)^2}\left[\ln^2 a_1 - \ln \overline{Z}_L \ln a_1 + \left(\frac{\ln \overline{Z}_L}{2\sin\beta\Delta}\right)^2\right].$$

Taking the derivative of ϵ with respect to ln a_1 and equating it to zero leads to $a_1 = \sqrt{\overline{Z}_L}$. The reflection coefficient for this value of a_1 is

$$\Gamma_i = \frac{\sin 2\beta\Delta}{4\beta\Delta} \ln \overline{Z}_L e^{-j2\beta\Delta}.$$

The error function is zero for $\Delta = n\lambda/4$ and $L = n\lambda/2$ for $n = 1, 2, 3, \cdots$. The comment at the end of Section II-B also holds here.

D. Piecewise Linear Approximation of $\ln \overline{Z}(z)$

The tapered line of length L is divided into N sections of width $\Delta = L/N$ as in Fig. 4. The division points are at $z_n = n\Delta$ for $n = 0, 1, 2, \cdots, N$. The value of $\ln \overline{Z}$ at z_n is taken as a_n and its variation in each section is assumed linear. Therefore

$$\ln \overline{Z}(z) = \frac{a_n - a_{n-1}}{\Delta} z + a_{n-1} - (n-1)(a_n - a_{n-1}),$$
$$(n-1)\Delta \le z \le n\Delta \quad n = 1, 2, 3, \cdots, N. \tag{31}$$

The boundary conditions at $z = 0$ (for $n = 1$ and $\overline{Z}(0) = 1$) and at $z = L$ (for $n = N$ and $\overline{Z}(L) = \overline{Z}_L$) are applied to get $a_o = 1$ and $a_N = \ln \overline{Z}_L$. Then substitutions of (31) in (1) or (2) yield (26) except that N is replaced by $N - 1$, since here the line is divided into N instead of $N + 1$ sections. Also note that the unknown quantities here are a_n, whereas in the previous section they are ln a_n. Consequently, the approximate operator for d/dz ln $\overline{Z}$ and the piecewise linear approximation of ln $\overline{Z}(z)$ are the same technique. Therefore, (26)–(30) apply here also, except as noted above N is replaced by $N - 1$. The characteristic impedance of each section is

$$\overline{Z}(z) = \exp\left[\frac{a_n - a_{n-1}}{\Delta} z + n a_{n-1} - (n-1)a_n\right]$$
$$\times (n-1)\Delta \le z \le n\Delta. \tag{32}$$

Alternately, the function ln $\overline{Z}(z)$ can be approximated by triangle functions of amplitude a_n as in Fig. 5

$$\ln \overline{Z} = \sum_{n=1}^{N} a_n T_n(z - n\Delta) \tag{33}$$

$$T_n(z) = \begin{cases} 1 - |z - n\Delta|\frac{1}{\Delta} & |z - n\Delta| < \Delta \\ 0 & |z - n\Delta| > \Delta \end{cases} \tag{34}$$

$$\frac{d}{dz}T_n(z) = \begin{cases} -\frac{1}{\Delta} & n\Delta < z < (n+1)\Delta \\ \frac{1}{\Delta} & (n-1)\Delta < z < n\Delta. \end{cases} \tag{35}$$

The triangle functions at the two ends of the line have only one linear segment, at $z = 0$ a line with a negative slope and

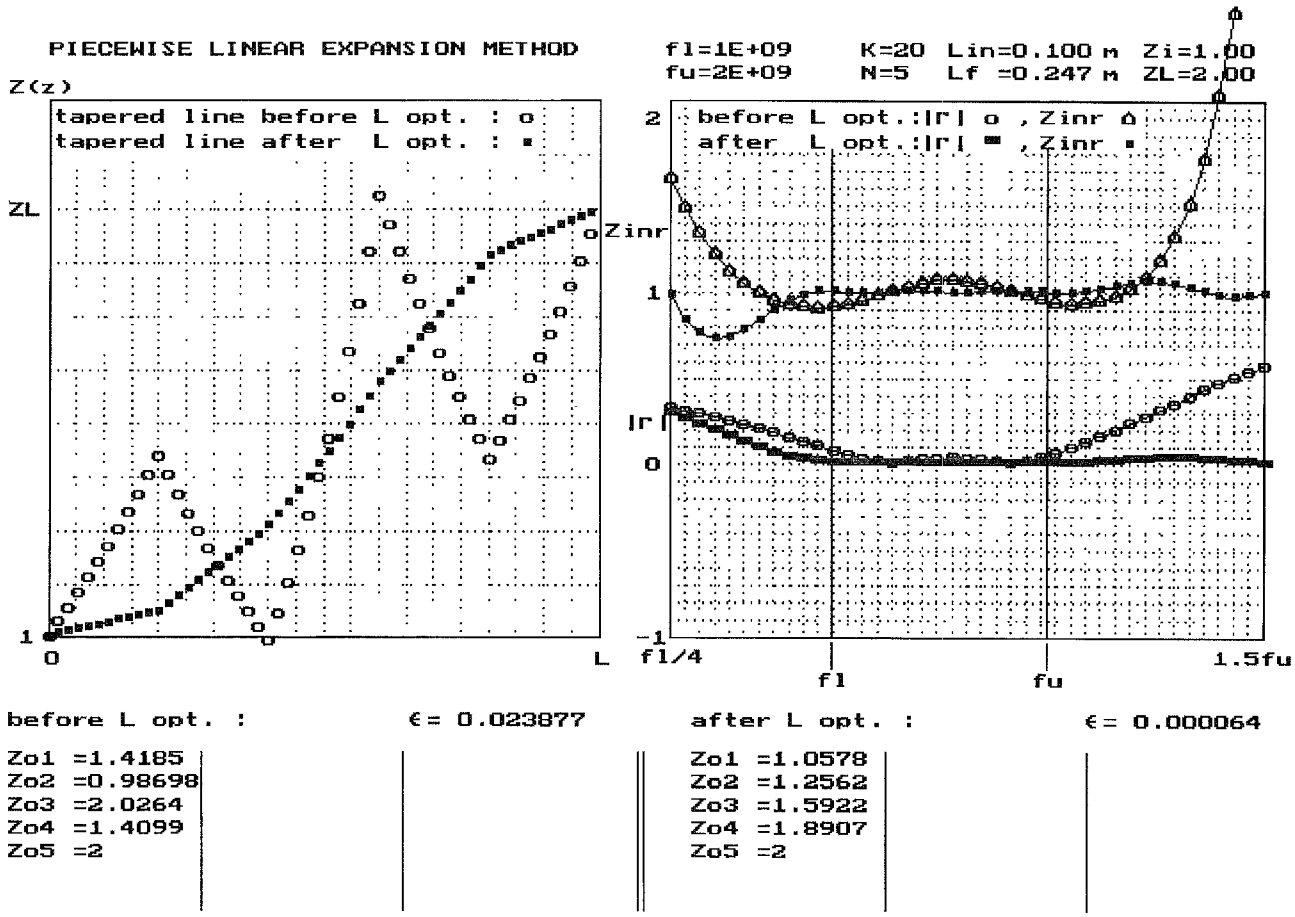

Fig. 10. Tapered line design by PLA.

at $z = L$ a line with a positive slope. Therefore, at $z = 0$ no triangle function is considered since $a_o = \ln \overline{Z}(0) = \ln 1 = 0$, and at $z = L$ only a positive slope line is considered since $a_N = \ln \overline{Z}(L) = \ln \overline{Z}_L$. Substitutions of (33)–(35) in (1) or (2) yield the same equation (26) with N replaced by $N - 1$.

We may also develop the piecewise linear approximation of $\overline{Z}(z)$. However, the expression of Γ_i will be in terms of cosine and sine integrals and the construction of the error function and its derivatives become unduly complicated.

For example, suppose the line length (L) has only one segment $(N = 1)$. Therefore, $\overline{Z}(z)$ in (31) and Γ_i in (26) with $N = 0$ are

$$\overline{Z}(z) = \exp\left(\frac{z}{L} \ln \overline{Z}_L\right)$$

$$\Gamma_i = \frac{1}{2}\frac{\sin \beta L}{\beta L} \ln \overline{Z}_L e^{-\beta L}.$$

The error is $\epsilon = |\Gamma_i|^2$. The condition for no reflection is $L = n\lambda/2$ for $n = 1, 2, 3, \cdots$. Compare this result with [2, p. 372].

For another example, suppose the line length (L) is divided into two segments $(N = 2)$ with width $\Delta = L/2$. The piecewise linear expansions in (31) are

$$\ln \overline{Z}(z) = \begin{cases} \frac{2}{L} a_1 z \\ \qquad 0 \le z \le L/2 \\ \frac{2}{L}(\ln \overline{Z}_L - a_1) z + 2a_1 - \ln \overline{Z}_L, \\ \qquad L/2 \le z \le L. \end{cases}$$

The reflection coefficient by (26) with $N = 1$ is

$$\Gamma_i = \frac{\sin(\pi L/\lambda)}{\pi L/\lambda} e^{-j(2\pi L/\lambda)} \cdot \left[ja_1 \sin(\pi L/\lambda) + \frac{1}{2} \ln \overline{Z}_L e^{-j(\pi L/\lambda)}\right].$$

The derivative of ϵ with respect to a_1 is equated to zero to obtain $a_1 = \ln \sqrt{\overline{Z}_L}$. Therefore

$$\overline{Z}(z) = \exp\left[\frac{z}{L} \ln \overline{Z}_L\right], \; 0 \le z \le L$$

$$\Gamma_i = \frac{1}{2} \ln \overline{Z}_L \frac{\sin(2\pi L/\lambda)}{(2\pi L/\lambda)} e^{-j(2\pi L/\lambda)}.$$

The error is $\epsilon = |\Gamma_i|^2$. The reflectionless line length is $L = n\lambda/2$. Compare this result with [2, p. 372].

III. Computer Implementation

Computer programs have been written for the four approximations of the characteristic impedance function of the transformer. Although the approximate operator (AO) and the piecewise linear approximation (PLA) essentially result in the same formulation, the former gives the impedances of a stepline (24), and the latter produces the impedance function of a taper (32), besides giving the stepline impedances. The pulse expansion (PE) formulation designs a stepline (23), and

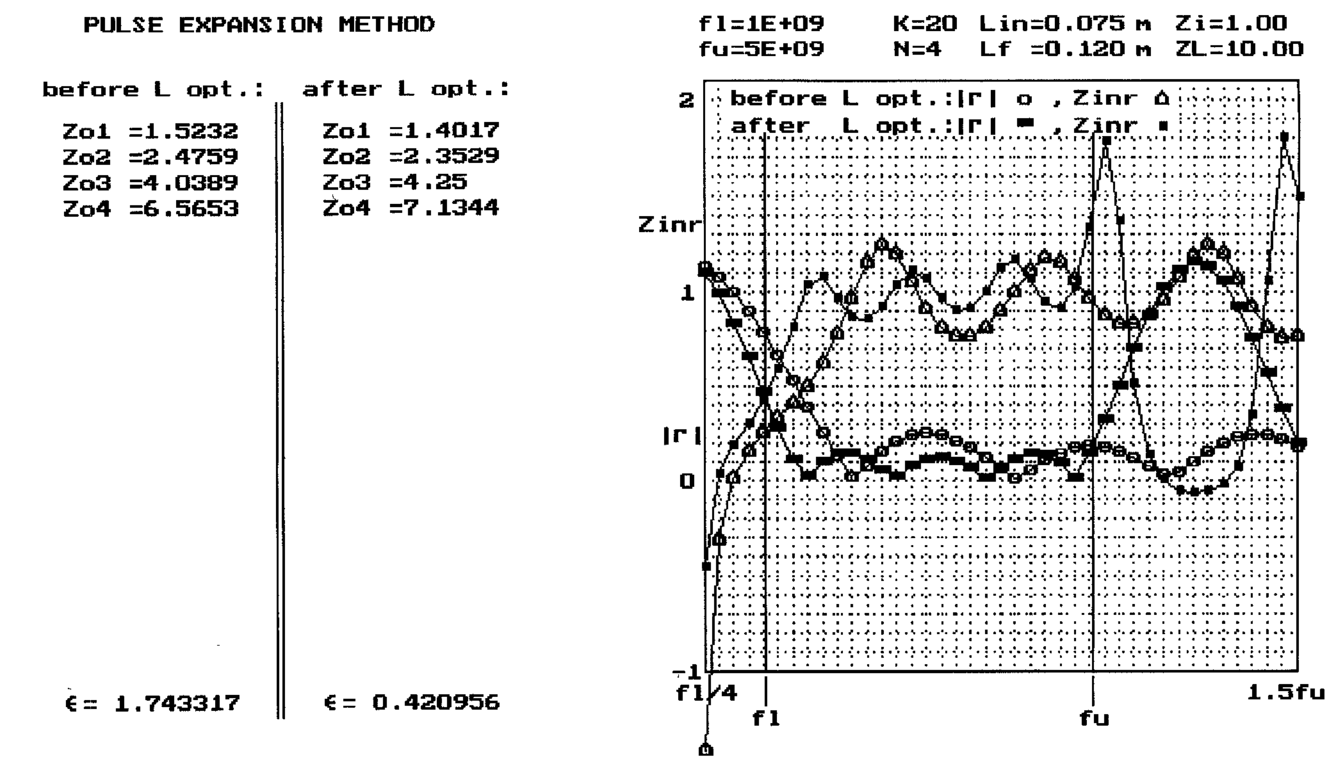

Fig. 11. Stepline design by PE.

the polynomial approximation (PA) gives the tapered line impedance function (7).

Several examples are given as illustrations of the applicability of MLS for the transformer designs, and the results are compared with the available data in the literature. The relevant data are summarized in Table I, wherein are given the initial line length (L_{in}), number of sections or variables (N), lower frequency (F_L) and upper frequency (F_u) limits of the desired matching bandwidth, number of frequencies to be matched ($K+1$), input impedance (Z_{in}), load impedance (Z_L), initial value of the error (ϵ_{in}) before the length optimization, final line length (L_f), and final value of the error (ϵ_f) after several iterations of the length optimization. The following formulas are used to calculate the value of the standing wave ratio SWR and the transformer input impedance $Z_{in} = R_{in} + jX_{in}$ in terms of the input reflection coefficient Γ_i in and around the desired frequency band, respectively

$$SWR = \frac{1+|\Gamma_i|}{1-|\Gamma_i|}, \overline{Z}_{in} = \frac{1+\Gamma_i}{1-\Gamma_i}.$$

The values of $|\Gamma_i|$, SWR, $\overline{R}_{in}$, and $\overline{X}_{in}$ may be drawn versus frequency. In the figures, however, the values of $|\Gamma_i|$ and R_{in} are drawn versus frequency before and after the length optimization, which shows the effect of the length optimization on the transformer design. The characteristic impedances of the stepline and/or the shape of the tapered line are also shown in the figures as the line characteristic impedance versus position on the line.

The programs first determine the transformer characteristic impedance function for an initial line length and then the error function is minimized with respect to the line length using the Newton's method

$$L_{i+1} = L_i - \frac{\epsilon_i}{\propto (\partial\epsilon/\partial L)_i}$$

where $\propto$ is a constant, chosen to speed up the computation. The linear search of interval halving can also be used [18]. The iterations are continued until the desired optimum transformer line length and characteristic impedance function is obtained.

Fig. 6 shows a tapered line design by PA to match a normalized load impedance of 2 Ω to an input impedance of 1 Ω. The initial line length is 5 cm, which is somewhat greater than a quarter wavelength of the lower frequency limit of the band. Good matching is achieved, but it improves after length optimization. Compare the result with [3, p. 323]. Fig. 7 shows a seven-section stepline design by PE to match 5–1 Ω. The initial line length is equal to a quarter wave of the lower frequency limit. Figs. 8 and 9 show the stepline and/or tapered line transformer design of the same example by the methods of AO and PLE, respectively. The results are quite the same. Note the zigzag variation of the stepline characteristic impedance for the shorter line lengths, which results in a bandpass filter behavior. As the line length increases as a result of the length optimization, the variation of the stepline or tapered line characteristic impedance becomes smooth, changing uniformly from about 1 to 5 Ω. This results in a high-pass filter behavior for the methods of AO and PLE. Such a behavior is more notable in Fig. 10, where a stepline is designed by PE to match 1–2 Ω in quite a narrow bandwidth of 1–2 GHz. Fig. 11 shows a four-section stepline design by PE to match 1–10 Ω. The initial line length is chosen equal to $\lambda_1/4$ = 7.5 (cm), which after the length optimization becomes 12 (cm). Compare this result with the Chebyshev transformer design in [3, p. 317].

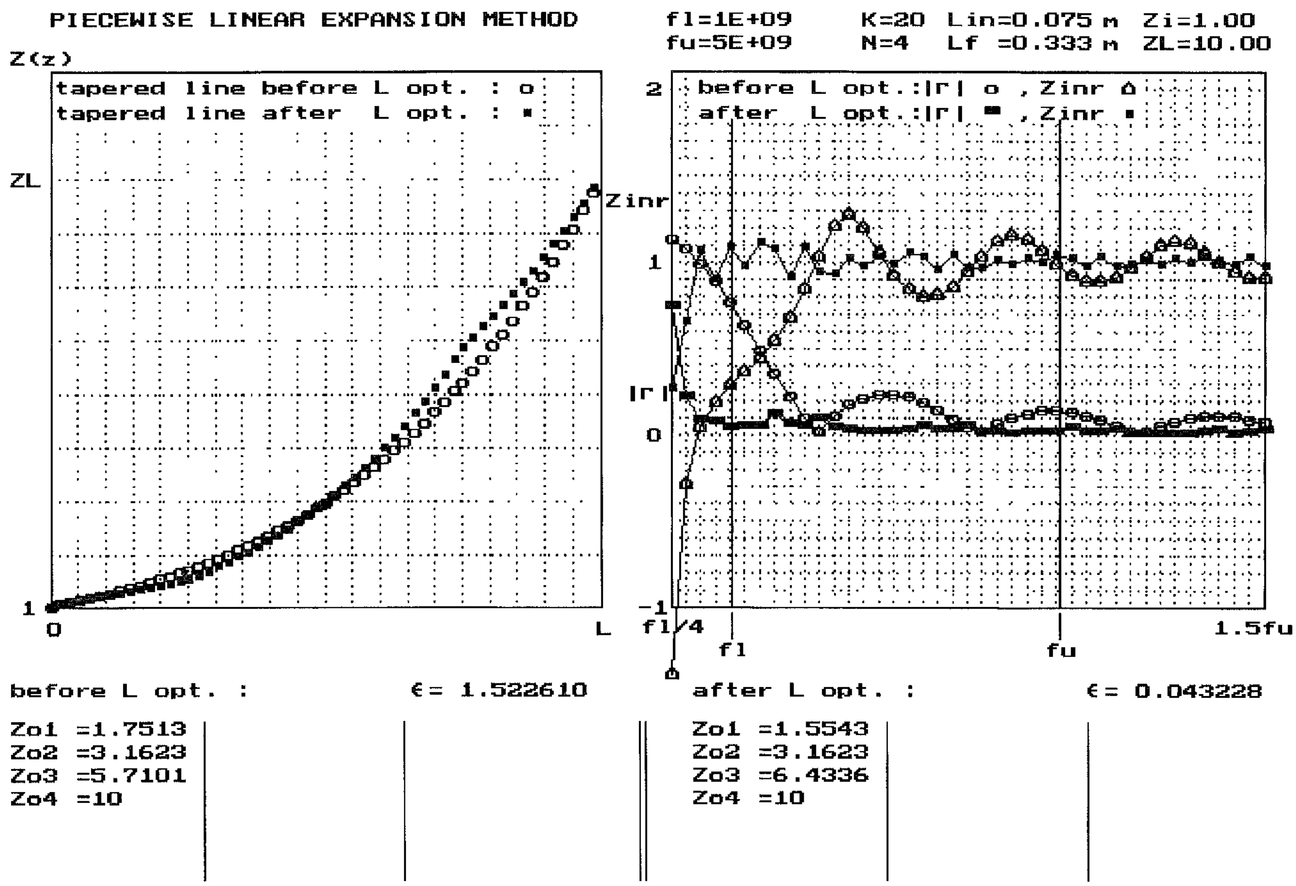

Fig. 12. Tapered line design by PLA.

Fig. 12 shows the transformer design of the same example by PLE, with better matching conditions.

It should be noted that the present formulation is based on the theory of small reflections. In order for the results to be reliable, the magnitude of the reflection coefficient should be small, say less than 0.1. In the figures the value of $|\Gamma_i|$ in some instances, particularly outside the specified frequency band, departs from zero, which makes the values of $|\Gamma_i|$, R_{in}, X_{in}, and SWR incorrect. In the examples presented, however, good impedance matching is achieved, since the value of $|\Gamma_i|$ is quite small in the specified frequency band and confirms the validity of the formulation based on (1). The results of the computer runs indicate that for the achievement of good impedance matching the line length should be about a quarter wavelength of the lower frequency limit of the bandwidth. Length optimization tends to increase the transformer length, resulting in better matching conditions. As the ratio of the output–input impedances increases, the required length of the transformer should increase to realize acceptable impedance matching. The transformer response curves in the figures show that the behavior of the tapered line is as a high-pass filter, whereas that of the stepline is as a bandpass filter. Therefore, increasing the upper frequency limit of the bandwidth does not appreciably change the tapered line shape and length.

Better impedance matching tends to increase the stepline length (L) and also tends to smooth up the variation of its characteristic impedance function (Z). In the case for a given stepline length, the characteristic impedance of the sections changes in large steps and has an erratic behavior resulting in poor matching conditions and then increasing the number of sections (N) gives an improved transformer design. But if the stepline length is too short for a given bandwidth, increasing the number of sections could not necessarily improve the impedance matching and the stepline length should be increased accordingly. Increasing the number of frequencies (K) improves the accuracy of the value of the error function (ϵ) and also increases the computation time. Increasing the number of stepline sections however, results in increased cpu time.

IV. Conclusion

It is proposed that the method of least squares as formulated here facilitates the design of impedance transformers, since a stepline or a tapered line of any length can be designed (from the viewpoint of minimizing the line input reflections) to match any two impedances over a frequency band. The designs of steplines and tapers are, to some extent, unified here.

The computer programs serve as a tool for designers to investigate the limitations of impedance matching for a set of initial conditions, such as the line length, frequency band, and input/output impedances. By varying the program input quantities, the engineer can arrive at an optimum design, taking into account limitations of space and realizability of the line characteristic impedances. Running the computer programs for various combinations of input values reveals interesting and uncommon stepline and taper shapes which for a given line length best match two specified impedances over a desired frequency band.

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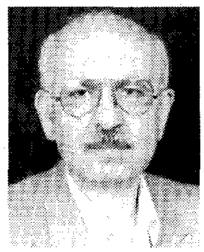

Homayoon Oraizi (S'69–M'90) was born on April 24, 1942, in Isfahan, Iran. He received the B.E.E. degree in 1967 from the American University of Beirut, Lebanon, and the M.Sc. and Ph.D. degrees in electrical engineering from Syracuse University, N.Y., in 1969 and 1973, respectively.

He is currently an Associate Professor and Head of the Communication Systems Group in the Electrical Engineering Department at Iran University of Science and Technology. From 1973 to 1974 he taught at Khaje-Nassir University, Tehran, Iran. From 1974 to 1985 he worked for Iran Electronics Industries, Shiraz, Iran, as a Systems Engineer, Supervisor, and Head of the Systems Engineering Department in the Communication Division engaged in various aspects of technology transfer mainly in the field of HF/VHF/UHF communication systems. In 1985, he joined the Electrical Engineering Department at the Iran University of Science and Technology, Tehran, as an Assistant Professor. His research interests are in the area of numerical methods in electromagnetics engineering. He teaches various electromagnetics engineering courses and has written two books on electromagnetics and fields and waves in Persian and has translated an antenna book into Persian. He has published several papers in international journals and conferences.